Parameter-free Locally Accelerated Conditional Gradients

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Problem Setting

Goal is L-smooth μ-strongly convex optimization over polytope $\mathcal{X}$.

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

Main ingredients:
First Order Oracle (FOO). Given $\mathbf{x} \in \mathcal{X}$ and a differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$, return:

$$\nabla f(\mathbf{x}) \in \mathbb{R}^n \text{ and } f(\mathbf{x}) \in \mathbb{R}$$

Linear Minimization Oracle (LMO). Given $\mathbf{v} \in \mathbb{R}^n$, return:

$$\operatorname*{argmin}_{\mathbf{x}\in\mathcal{X}} \langle \mathbf{v}, \mathbf{x} \rangle$$

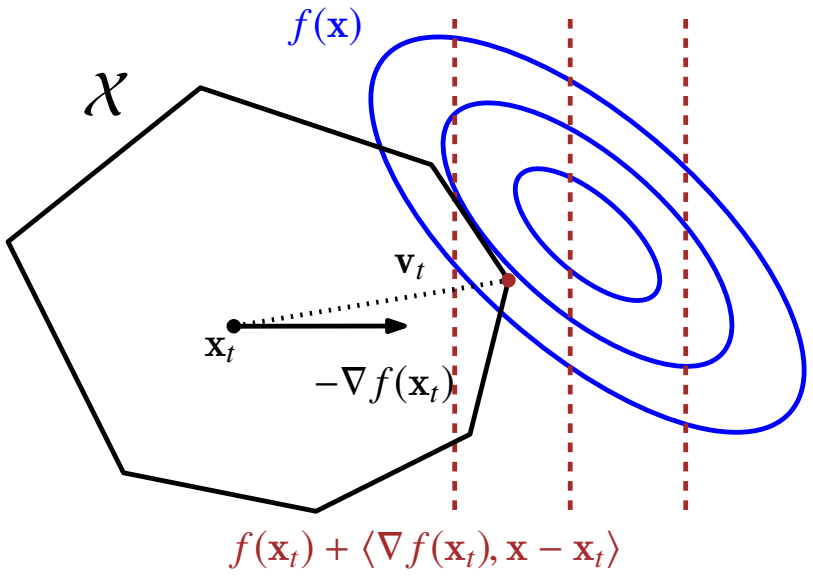
Conditional Gradients

Focus on the *Conditional Gradients/Frank-Wolfe* algorithm [FW56; Pol74] and its variants. The basic variant is:

CG Algorithm.

Input: $\mathbf{x}_0 \in \mathcal{X}$, stepsizes $\gamma_t \in (0, 1]$.

1: **for** $t = 0$ to T **do**
2: $\quad \mathbf{v}_t = \operatorname{argmin}_{\mathbf{x} \in \mathcal{X}} \langle \nabla f(\mathbf{x}_t), \mathbf{x} \rangle$
3: $\quad \mathbf{x}_{t+1} = \mathbf{x}_t + \gamma_t (\mathbf{v}_t - \mathbf{x}_t)$
4: **end for**



Optimal Complexity

Optimal **projection-based** methods for this class of functions achieve an ϵ-optimal in $T = O\left(\sqrt{\frac{L}{\mu}} \log \frac{1}{\epsilon}\right)$ **first-order calls** [NY83; Nes83].

Q: Can CG-type methods have the same oracle complexity, both in terms of FOO to f and LMO to $\mathcal{X}$?

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Yes! Albeit locally [DCP20] if we know L and μ.

Contributions

The contributions are as follows:

1. Parameter-free Locally-accelerated Conditional Gradient (PF-LaCG) algorithm.
2. Near-optimal and parameter-free accelerated algorithm (ACC) with inexact projections.

Main Ideas

The PF-LaCG algorithm [Car+21] couples the *Away-step Frank-Wolfe* (AFW) [GM86; LJ15] and ACC, a variant of the accelerated algorithm [CDO18], and periodically restarts when a measure of optimality is halved, and in this case **an upper bound on the primal gap**, denoted as:

$$w(\mathbf{x}, \mathcal{S}) \stackrel{\text{def}}{=} \max_{\mathbf{u} \in \mathcal{X}, \mathbf{v} \in \mathcal{S}} \langle \nabla f(\mathbf{x}), \mathbf{u} - \mathbf{v} \rangle .$$

where $\mathcal{S}$ is a proper support [1] of $\mathbf{x}$.

[1] A support $\mathcal{S}$ of $\mathbf{x}$ is a proper support of $\mathbf{x}$ if the weights associated with the convex decomposition are positive.

Main Idea

$$w(\mathbf{x}, \mathcal{S}) \stackrel{\text{def}}{=} \max_{\mathbf{u} \in \mathcal{X}, \mathbf{v} \in \mathcal{S}} \langle \nabla f(\mathbf{x}), \mathbf{u} - \mathbf{v} \rangle .$$

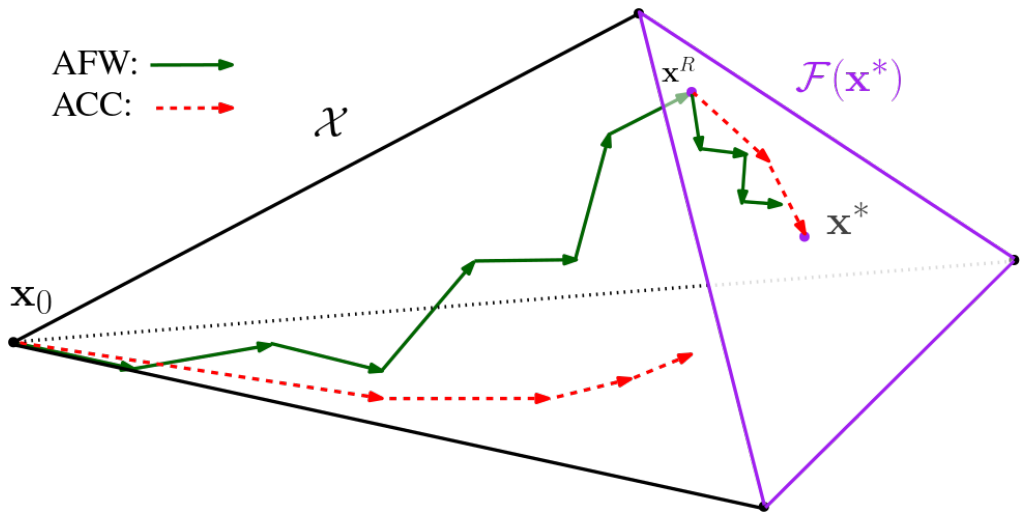
This allows us to:

- Maintain a computable global measure of optimality without knowing L and μ.
- Couple the AFW algorithm and the ACC algorithm while guaranteeing monotonic progress

Algorithm Sketch

At a high level:

1. PF-LaCG runs AFW and ACC in parallel, and restarts every time AFW halves $w(\mathbf{x}, \mathcal{S})$. After every restart choose point with lower value of $w(\mathbf{x}, \mathcal{S})$ and potentially update active set of ACC
2. After a finite number of iterations independent of ϵ, the active set of AFW contains $\mathbf{x}^*$ **and** ACC converges to the optimal at an accelerated rate



Convergence Rate of PF-LaCG

Theorem (Convergence rate of PF-LaCG)

Let f be L-smooth and μ-strongly convex. The number of calls to FOO and LMO required to reach an ϵ-optimal solution, measured in terms of $w(\mathbf{x}, \mathcal{S})$, to the minimization problem satisfies:

$$T = \min \Bigg\{ \underbrace{O\left(\frac{LD^2}{\mu\delta^2} \log \frac{1}{\epsilon}\right)}_{\text{AFW bound}},$$

$$\underbrace{K}_{\text{Burn-in}} + \underbrace{O\left(\sqrt{\frac{L}{\mu}} \log\left(\frac{L}{\mu}\right) \log\left(\frac{LD}{\mu\delta}\right) \log \frac{1}{\epsilon}\right)}_{\text{Locally-accelerated convergence}} \Bigg\},$$

where K is a constant that is independent of ϵ.

Computational Results

Structured LASSO Regression[2]

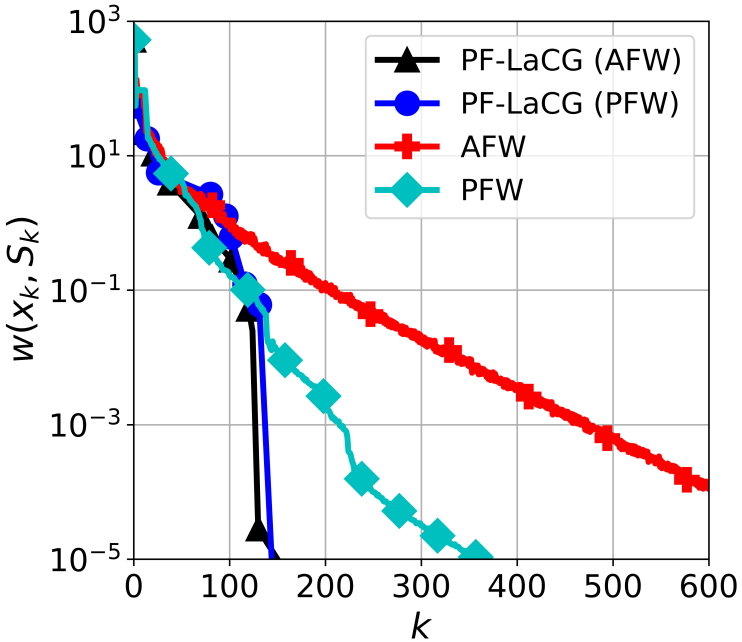


Figure: $w(\mathbf{x}_t, \mathcal{S}_t)$ vs. iteration

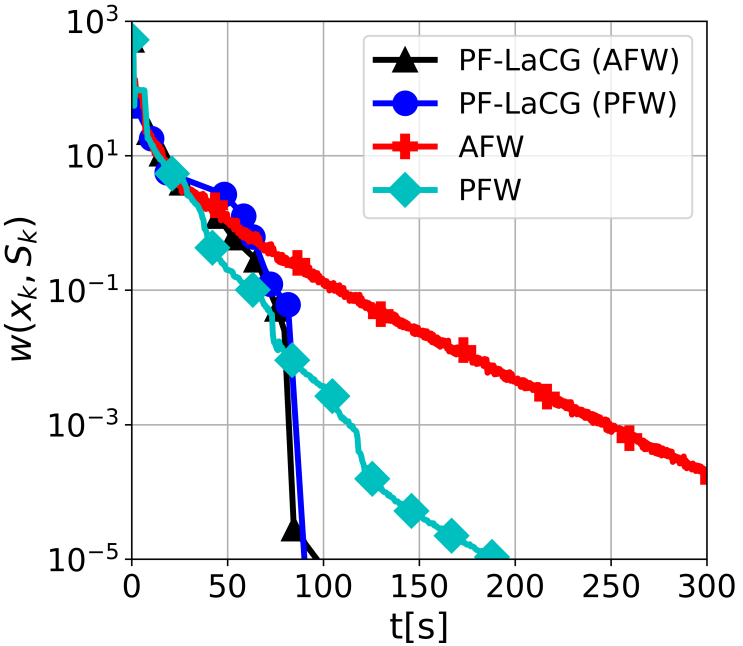


Figure: $w(\mathbf{x}_t, \mathcal{S}_t)$ vs. time

[2]Note: Other measures of optimality can be bounded above by $w(\mathbf{x}, \mathcal{S})$.

Summary

- We introduce PF-LaCG, a novel parameter-free projection-free algorithm for minimizing smooth and strongly convex functions over convex polytopes.
- After a finite burn-in phase independent of ϵ, PF-LaCG achieves a near-optimal accelerated convergence rate without knowledge of any problem parameters.
- We demonstrate PF-LaCG's practical improvements over non-accelerated algorithms, both in iteration count and wall-clock time.

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